

Topic 3 -

Linear first order

ODEs



A linear first order ODE is an equation of the form

$$a_1(x)y' + a_0(x)y = g(x)$$

If we are considering an interval I where $a_1(x) \neq 0$ for any x in I then we can divide through by $a_1(x)$ to get

$$y' + a(x)y = b(x)$$

$$\text{where } a(x) = \frac{a_0(x)}{a_1(x)} \text{ and } b(x) = \frac{g(x)}{a_1(x)}.$$

This is the type of equation that we will consider for now.

Suppose we have a linear first order ODE of the form

$$y' + a(x)y = b(x) \quad (*)$$

where $a(x)$ and $b(x)$ are continuous on an open interval I .

Let's solve this.

Suppose $\phi(x)$ solves $(*)$ on I .

That is,

$$\phi'(x) + a(x)\phi(x) = b(x) \quad (**)$$

for all x in I .

Let $A(x)$ be an anti-derivative of $a(x)$ on I .

Note: $A(x)$ exists by the FTC since $a(x)$ is continuous.

Ex:

$$y' + \underbrace{2x}_a y = \underbrace{x}_b$$

$a(x) = 2x$ $b(x) = x$

$$I = (-\infty, \infty)$$

Ex:

$$a(x) = 2x$$
$$A(x) = x^2$$

Multiply $(*)$ by $e^{A(x)}$ to get:

$$e^{A(x)} \phi'(x) + a(x)\phi(x) = e^{A(x)} b(x)$$

This gives (by the product rule $(fg)' = f'g + g'f$)

$$\left(e^{A(x)} \phi(x) \right)' = e^{A(x)} b(x)$$

Let $B(x)$ be an anti-derivative of $e^{A(x)} b(x)$ on I .

Then ϕ solves $(*)$ on I if and only if

$$e^{A(x)} \phi(x) = B(x) + C$$

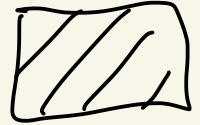
where C is some constant.

So, ϕ solves $(*)$ on I if and only if

$$\phi(x) = B(x)e^{-A(x)} + Ce^{-A(x)}$$

integrate both sides w) respect to x

Since all the steps above
are reversible since $e^{A(x)} \neq 0$
we know we have found
the general solution to (*).



Ex: Solve

$$y' + \underbrace{2xy}_{a(x)=2x} = \underbrace{x}_{b(x)=x}$$

on $I = (-\infty, \infty)$.

Step 1: $A(x) = \int 2x dx = x^2$

Multiply by $e^{A(x)} = e^{x^2}$ to get

$$e^{x^2} y'(x) + 2x e^{x^2} y(x) = x e^{x^2}$$

Step 2: Undo the product rule on the left-hand side:

$$(e^{x^2} y(x))' = x e^{x^2}$$

Step 3: Integrate both sides to get:

$$e^{x^2} y(x) = \underbrace{\frac{1}{2} e^{x^2} + C}$$

↑

$$\int x e^{x^2} dx = \frac{1}{2} \int e^u du$$

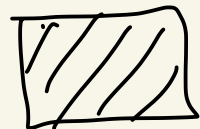
$$\begin{aligned} u &= x^2 \\ du &= 2x dx \\ \frac{1}{2} du &= x dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{x^2} + C \end{aligned}$$

Step 4:

Thus, $y = \frac{1}{2} + C e^{-x^2}$

for some constant C .




Ex: Let's solve

$$y' + \underbrace{\cos(x)}_{a(x)=\cos(x)} y = \underbrace{\sin(x)\cos(x)}_{b(x)=\sin(x)\cos(x)}$$

on $I = (-\infty, \infty)$

Step 1: Let $A(x) = \sin(x)$.

Then $A'(x) = \cos(x) = a(x)$ 

Multiply by $e^{A(x)} = e^{\sin(x)}$.

We get

$$e^{\sin(x)} y'(x) + \cos(x) e^{\sin(x)} y(x) = \sin(x)\cos(x) e^{\sin(x)}$$

Step 2: Undo the product rule
on the left side to get:

$$\left(e^{\sin(x)} y(x) \right)' = \sin(x)\cos(x) e^{\sin(x)}$$

Step 3: Integrate both sides.

$$e^{\sin(x)} y(x) = \underbrace{\sin(x) e^{\sin(x)} - e^{\sin(x)}}_{\uparrow} + C$$

$$\int \sin(x) \cos(x) e^{\sin(x)} dx$$

$$= \int t e^t dt = t e^t - \int e^t dt$$

$$\begin{aligned} t &= \sin(x) \\ dt &= \cos(x) dx \end{aligned}$$

$$\begin{aligned} u &= t & dv &= e^t dt \\ du &= dt & v &= e^t \end{aligned}$$

$$\int u dv = uv - \int v du$$

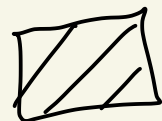
$$= t e^t - e^t + C$$

$$= \sin(x) e^{\sin(x)} - e^{\sin(x)} + C$$

Step 4: Thus,

$$y = \sin(x) - 1 + C e^{-\sin(x)}$$

where C is some constant.



Ex: Consider the equation

$$x y' + y = 3x^3 + 1$$

on $I = (0, \infty)$.

Since $x \neq 0$ on I we can divide by x to get

$$y' + \underbrace{\frac{1}{x}}_{a(x)} y = \underbrace{3x^2 + \frac{1}{x}}_{b(x)}$$

Step 1: Let $A(x) = \ln(x)$.

Then, $A'(x) = \frac{1}{x}$ for all x in I .

Multiply by $e^{A(x)} = e^{\ln(x)} = x$ to get

$$x y' + y = 3x^3 + 1$$

Step 2: Undo the product rule to get

$$(x y)' = 3x^3 + 1$$

Step 3: Integrate both sides to get

$$xy = \int (3x^3 + 1) dx = \frac{3}{4}x^4 + x + C$$

Step 4: Thus,

$$y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$$

where C is a constant.

Ex: Solve

$$x y' + y = 3x^3 + 1$$

$$y(1) = 2$$

on $I = (0, \infty)$

From above we know $y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$.

Plugging in $y(1) = 2$ we get

$$2 = y(1) = \frac{3}{4}(1)^3 + 1 + \frac{C}{1}$$

So,

$$2 = \frac{7}{4} + C$$

Thus,

$$C = \frac{1}{4}$$

Thus,

$$y = \frac{3}{4}x^3 + 1 + \frac{C}{x}$$